Finite-size scaling and the toroidal partition function of the critical asymmetric six-vertex model

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Finite-size corrections to the energy levels of the asymmetric six-vertex model transfer matrix are considered using the Bethe ansatz solution for the critical region. The nonuniversal complex anisotropy factor is related to the bulk susceptibilities. The universal Gaussian coupling constant g is also related to the bulk susceptibilities as $g=2H^{-1/2}/\pi$, H being the Hessian of the bulk free energy surface viewed as a function of the two fields. The modular covariant toroidal partition function is derived in the form of the modified Coulombic partition function which embodies the effect of incommensurability through two mismatch parameters. The effect of twisted boundary conditions is also considered.

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I. INTRODUCTION

The six-vertex model was first introduced as a model for the residual entropy of ice and for related ferroelectric transitions [1] but more recently, several other physical applications are being found. In the body-centered solid-on-solid models, the six-vertex configuration is mapped to the surface configuration of the fcc and bcc faces and the free energy of the six-vertex model as a function of the horizontal and vertical fields depicts the equilibrium crystal shapes (ECS) [2,3]. Also, on the stochastic surface where the vertex weights satisfy a certain relation, the transfer matrix of the six-vertex model can be regarded as the transition matrix of probabilistic cellular automata describing the dynamics of a driven lattice gas system and a $(1+1)$-dimensional surface growth model [4–6]. The five-vertex model, which is a special asymmetric limit of the six-vertex model, can be viewed as a model for an interacting domain wall system [7], interacting dimers [8], and certain types of crystal surfaces [9]. In these applications, it is necessary to consider the asymmetric six-vertex (ASSV) model in which there are nonzero horizontal and vertical fields which break the arrow reversal symmetries. It becomes the symmetric six-vertex (SSV) model when the fields are zero. The Heisenberg XXZ chain is a closely related problem. The Hamiltonian for the XXZ chain can be obtained from an anisotropic limit of the transfer matrix of the six-vertex model [6]. The vertical field corresponds to the magnetic field which couples to the z component of spins and the horizontal field corresponds to the asymmetry in the two hopping rates. The XXZ chain is called symmetric (asymmetric) if the two hopping rates are the same (different).

The phase diagram and the nature of the phase transition in the ASSV model are well known [1,10]. The phase diagram consists of the ferroelectrically and antiferroelectrically ordered phases and the disordered phase. In particular, the disordered phase shows interesting scaling properties. It is the critical phase with continuously varying critical exponents that is described by the central charge $c=1$ conformal field theory (CFT) in the continuum limit [11]. Since the $c=1$ theory plays a basic role in the theory of two-dimensional critical phenomena, there have been many works on the scaling properties of the six-vertex model. However, most of these works are confined to the SSV model [12] or the symmetric XXZ chain with or without the magnetic field [13–16] and to the special case of the five-vertex model [7], while the critical properties of the general ASSV model are also of interest. One purpose of this work is to fill this gap. In this paper we investigate the critical properties of the ASSV model through the finite-size scaling (FSS) studies on the transfer-matrix spectra. Our results can also be applied to the case of the asymmetric XXZ chain.

The FSS is very useful in studying the critical behaviors of two-dimensional statistical systems on lattice [17]. When the system is in a critical phase which possesses the conformal symmetry, it imposes strong restrictions on the FSS form of the eigenvalue spectra of the transfer matrix. Let $\mathbf{T}$ denote the row-to-row transfer matrix for a system with N columns and M rows. We assume periodic boundary conditions unless stated otherwise. The energy of the level α defined by $E_\alpha=-\ln\Lambda_\alpha$, Λ_α being the αth eigenvalue of $\mathbf{T}$, is expected to follow the scaling form

$$E_\alpha=Nf+\frac{2\pi}{N}\zeta''\left(h_\alpha+\bar{h}_\alpha-\frac{c}{12}\right)-\frac{2\pi i}{N}\zeta'(h_\alpha-\bar{h}_\alpha)+o\left(\frac{1}{N}\right), \tag{1}$$

where f is the bulk free energy (in units of k_BT), ζ' (ζ'') is the real (imaginary) part of the complex anisotropy factor $\zeta=\zeta'+i\zeta''$, c is the central charge, $(h_\alpha,\bar{h}_\alpha)$ are the conformal dimensions associated with the level, and $o(x)$ stands for terms smaller than x; $\lim_{x\to 0}o(x)/x=0$. The level whose energy follows the scaling form of Eq. (1) is associated with a primary operator and its descendants of the corresponding CFT. When the one-lattice-unit translation by $\mathbf{T}$ is incommensurate with the periodicity of the underlying model, an imaginary term of $O(1)$ may appear in the right-hand side of Eq. (1) [18]. The toroidal partition function (TPF) $\tilde{\mathcal{Z}}$ is defined as the $O(1)$ part of the partition function:

$$\tilde{\mathcal{Z}}=\lim_{\substack{N,M\to\infty\\ M/N=\text{fixed}}}\sum_\alpha e^{-M(E_\alpha-Nf)}, \tag{2}$$

where the sum is over all levels. If we use the scaling form of Eq. (1), it takes the form

$$\tilde{\mathscr{Z}}=(\mathsf{q}\bar{\mathsf{q}})^{-c/24}\sum_{\alpha}\mathsf{q}^{h_\alpha}\bar{\mathsf{q}}^{\bar{h}_\alpha}, \tag{3}$$

where the nome $\mathsf{q}\equiv e^{2\pi i\tau}$ with $\tau=\tau'+i\tau''=(M/N)\zeta$ and $\bar{\mathsf{q}}$ is the complex conjugate of q. The complex parameter τ is the modular ratio of the torus on which the corresponding CFT is defined and specifies how the $N\times M$ lattice should be deformed to make the system isotropic in the continuum limit. The TPF contains complete information on the spectra or the operator content of the model and enjoys the modular invariance (or more generally, modular covariance) properties which together with the conformal invariance principle are often sufficient to determine the form of $\tilde{\mathscr{Z}}$ [19]. The $c=1$ CFT consists of three isolated points and two one-parameter families [20]. The latter two describe the critical eight-vertex model or the Ashkin-Teller model and the SSV model or the Gaussian model compactified on a circle, respectively. They are related by a duality transformation. The TPF of the $c=1$ theory corresponding to the SSV line is the so-called Coulombic partition function [12] given by, for M and N even,

$$\tilde{\mathscr{Z}}_C=\frac{1}{|\eta(\mathsf{q})|^2}\sum_{m,n\in\mathbf{Z}}\mathsf{q}^{\Delta_{m,n}}\bar{\mathsf{q}}^{\bar{\Delta}_{m,n}}, \tag{4}$$

where $\eta(\mathsf{q})$ is the Dedekind eta function,

$$\eta(\mathsf{q})=\mathsf{q}^{1/24}\prod_{n=1}^{\infty}(1-\mathsf{q}^n), \tag{5}$$

and the conformal dimensions are given by

$$\Delta_{m,n}=\frac{1}{4}\left(\frac{m}{\sqrt{g}}+\sqrt{g}n\right)^2,\qquad \bar{\Delta}_{m,n}=\frac{1}{4}\left(\frac{m}{\sqrt{g}}-\sqrt{g}n\right)^2. \tag{6}$$

Here g is the Gaussian coupling constant and the integer indices m and n label the spin-wave excitation and the vortex excitation, respectively. The Gaussian coupling constant g is defined in such a way that it takes the value 1/2 for the free fermion theory and is related to K_R of Ref. [21] by $g=2\pi K_R$. However, the CFT being a bootstrap theory, it does not tell us how g and τ are related to the lattice model parameters. To obtain that information, one needs to rely on the FSS analysis.

The transfer matrix for the six-vertex model or equivalently the Hamiltonian of the XXZ chain is diagonalized [1] by the Bethe ansatz method for general boundary conditions. In this paper we present a method of calculating the finite-size corrections of arbitrary low-lying energies in the whole parameter space of the ASSV model. Starting from the Bethe ansatz solution for the system of width N, we derive a systematic expansion in $1/N$ of the energies assuming certain analyticity properties of the phase function which is introduced in Sec. II. The method is very similar to that used in [15] which considered the symmetric XXZ chain in a magnetic field but is generalized to be applicable to the general cases. From the expansion, we find that the whole critical phase of the ASSV model is also in the Gaussian-model universality class with $c=1$, where the coupling constant g is obtained from the solution of an integral equation. We also find that though the susceptibilities are nonuniversal, i.e., they depend on nonuniversal parameters, the Hessian of the free energy is simply given by

$$H\equiv\begin{vmatrix}\dfrac{\partial^2 f}{\partial^2 h} & \dfrac{\partial^2 f}{\partial h\,\partial v}\\[2ex] \dfrac{\partial^2 f}{\partial v\,\partial h} & \dfrac{\partial^2 f}{\partial^2 v}\end{vmatrix}=\left(\frac{2}{\pi g}\right)^2 \tag{7}$$

and hence depends on the model parameters only through g. Here, h and v are the horizontal and vertical fields, respectively, in units of k_BT. Since we find the FSS form for a general class of energy levels, we are able to construct the TPF for the ASSV model with the periodic boundary conditions explicitly. The resulting expression is given by the modified Coulombic partition function

$$\tilde{\mathscr{Z}}=\frac{1}{|\eta(\mathsf{q})|^2}\sum_{m,n\in\mathbf{Z}}e^{-2\pi i m\alpha}\mathsf{q}^{\Delta_{m,n-\beta}}\bar{\mathsf{q}}^{\bar{\Delta}_{m,n-\beta}}, \tag{8}$$

where α and β as defined in Eq. (72) are the two mismatch parameters which account for the incommensurability of the lattice with the mean distances between down arrows and left arrows, respectively. The ASSV model under the twisted boundary conditions (explained in Sec. II) is equivalent to the ASSV model under the periodic boundary conditions with modified fields. Using this relation the TPF for the ASSV model under the twisted boundary conditions is also derived, which confirms the conjectured TPF for the SSV model under the twisted boundary condition in the horizontal direction [22].

This paper is organized as follows. In Sec. II we give a brief review of the transfer-matrix formulation and its Bethe ansatz solution of the ASSV model. The classification scheme of the Bethe ansatz solutions is presented. The relation between the ASSV model under the periodic and the twisted boundary conditions is also discussed. In Sec. III, under a certain assumption we derive a summation formula which converts a sum over functions of fugacities for general levels into a finite-size expansion. The assumption turns out to be the sufficient condition for the criticality in Sec. IV. We obtain an integral equation for the expansion coefficients and find that the finite-size correction terms are related to the partial derivatives of the bulk term with respect to the horizontal and vertical fields. Some of the details of calculations are relegated to Appendix A. In Sec. IV, applying the summation formula to the transfer-matrix eigenvalues, we derive the connections to the $c=1$ CFT together with the expressions for g. Since we find the FSS form for the general scaling levels, the TPF's for the periodic and the general twisted boundary conditions are derived. In Sec. V we summarize our results and discuss relations to other works. Also, possible physical relevances to the ECS are discussed. In Appendix B we prove an identity $J=1$ where J appears in the relations between the finite-size correction terms and susceptibilities. In Appendix C the modular transformation

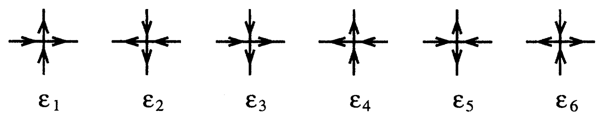

FIG. 1. Six-vertex configurations.

properties of the TPF for the ASSV model with general boundary conditions are discussed.

II. THE ASYMMETRIC SIX-VERTEX MODEL

On the square lattice of N columns and of M rows, the six-vertex model configurations are obtained by covering the bonds of the square lattice with arrows which satisfy the ice rule: At each vertex there are two arrows in and two arrows out. The six-vertex configurations satisfying the ice rule are shown in Fig. 1. Following the notation of Ref. [10], the vertex energies are assigned as

$$\varepsilon_1=-\frac{\delta}{2}-h-v,\quad \varepsilon_2=-\frac{\delta}{2}+h+v,$$

$$\varepsilon_3=\frac{\delta}{2}-h+v,\quad \varepsilon_4=\frac{\delta}{2}+h-v,\tag{9}$$

$$\varepsilon_5=-\epsilon,\quad \varepsilon_6=-\epsilon.$$

We write the energy in units of k_BT and denote the vertex weights as $w_i=\exp(-\varepsilon_i)$. $h(v)$ is the horizontal (vertical) electric field which is conjugate to the horizontal (vertical) polarization $1-2\rho_L$ $(1-2\rho_D)$ where ρ_L (ρ_D) is the mean density of left (down) arrows. For later uses, we define parameters

$$\Delta=\frac{w_1w_2+w_3w_4-w_5w_6}{2\sqrt{w_1w_2w_3w_4}}=\frac{e^{\delta}+e^{-\delta}-e^{2\epsilon}}{2}\tag{10}$$

and

$$\tilde{\Delta}=\frac{w_1w_2+w_3w_4-w_5w_6}{w_1w_3+w_2w_4}=\frac{\Delta}{\cosh(2h)}.\tag{11}$$

The partition function $\mathscr{Z}$ is written as

$$\mathscr{Z}=\mathrm{Tr}\mathbf{T}^M,\tag{12}$$

where $\mathbf{T}$ is the row-to-row transfer matrix whose definition can be found in Ref. [10]. Let Q denote the number of down arrows per row and let

$$q=\frac{Q}{N}.\tag{13}$$

The thermal average of q is ρ_D. Since Q must be the same in the two adjacent rows of vertical bonds due to the ice rule, $\mathbf{T}$ can be considered in each subspace of fixed number of Q, which will be called the Q sector, separately. From the Bethe ansatz method [1], the eigenvalue of $\mathbf{T}$ in the Q sector is given by

$$\Lambda=\Lambda_R+\Lambda_L,\tag{14a}$$

where

$$\Lambda_R=w_1^{N-Q}\prod_{j=1}^{Q}\left(w_3-\frac{w_5w_6}{w_4-w_1z_j}\right),\tag{14b}$$

$$\Lambda_L=w_4^{N-Q}\prod_{j=1}^{Q}\left(w_2+\frac{w_5w_6z_j}{w_4-w_1z_j}\right),\tag{14c}$$

where the fugacities $\{z_j\}$ are given by the solutions of the Bethe ansatz equation

$$z_i^N=(-1)^{Q-1}\prod_{j=1}^{Q}\frac{1+e^{4h}z_iz_j-2e^{2h}\Delta z_i}{1+e^{4h}z_iz_j-2e^{2h}\Delta z_j}.\tag{15}$$

Note that the fugacities z_j introduced in Eqs. (14) and (15) are the inverses of the fugacities used in [6] and that the five-vertex model is achieved in the limit $h\to\pm\infty$ keeping $\tilde{\Delta}$ fixed. Introducing a variable p such that $e^{ip}=ze^{2h}$ and the phase function

$$Z_N^0(p)=p+2hi-\frac{1}{N}\sum_{j=1}^{Q}\Theta^0(p,p_j),\tag{16}$$

where

$$e^{i\Theta^0(p,p')}\equiv\frac{1+e^{i(p+p')}-2\Delta e^{ip}}{1+e^{i(p+p')}-2\Delta e^{ip'}},\tag{17}$$

the Bethe ansatz equation takes the simple form

$$Z_N^0(p_j)=\frac{2\pi I_j}{N}\quad(j=1,\ldots,Q),\tag{18}$$

where I_j are half integers (integers) for Q even (odd) with range $-N/2<I_j\leq N/2$. It is obtained by taking the logarithm of Eq. (15) with the substitution of z_je^{2h} by e^{ip_j} and the quantity I_j arises from the choice of a phase factor $(-1)^{Q-1}=\exp[\pi i(2\times\text{integer}+\mathrm{Q}-1)]$. If we define the energies E_Q^R and E_Q^L through the relation

$$\Lambda_{R,L}=\exp[v(N-2Q)-E_Q^{R,L}],\tag{19}$$

they are given as

$$E_Q^{R,L}=-\left[\pm\left(h+\frac{\delta}{2}\right)N+\sum_{j=1}^{Q}\Phi_{R,L}^0(p_j)\right],\tag{20}$$

where

$$\Phi_{R,L}^0(p)=\ln\frac{2\Delta-e^{\pm\delta}-e^{\pm ip}}{1-e^{\pm(\delta+ip)}},\tag{21}$$

and $\{p_j\}$ are the distinct solutions of Eq. (18). In Eqs. (20) and (21), the upper (lower) sign corresponds to the R (L) case.

The bulk free energy is obtained from the ground-state energy. Using the ground-state energy, the free energy e as a function of h and q is given as

$$e(h,q)=\lim_{N,Q\to\infty}\min_{R,L}\left\{\frac{E_Q^R}{N},\frac{E_Q^L}{N}\right\}, \qquad (22)$$

where the limit is taken with fixed q. The free energy f as a function of h and v is given by the Legendre transformation of e, i.e.,

$$f(h,v)=\min_{0\le q\le 1}\{e(h,q)-(1-2q)v\}. \qquad (23)$$

The mean down-arrow density ρ_D is the value of q which minimizes the expression above while the left-arrow density ρ_L is given by $\rho_L=(1+\partial e/\partial h)/2$. Of the four model parameters, only Δ and h enter the Bethe ansatz equation determining $\{p_i\}$ while $\Phi^0_{R,L}$ depends only on Δ and δ. The vertical field v simply adjusts the mean value of q. Note that the horizontal field h plays the role of an additive constant to the phase function in Eq. (16). This fact enables one to follow the same line of analysis as in [15]. The asymmetric XXZ Hamiltonian is diagonalized with the same Bethe ansatz method but with a different energy function [6]. In the next section we will show that the FSS properties do not depend on the form of the energy function. So the asymmetric XXZ chain shares the same critical property with the ASSV model.

Different choices of the set $\{I_j\}$ in Eq. (18) lead to different eigenstates. It is well established that the ground-state energy is obtained if $I_j=-(Q+1)/2+j$. In analogy with the free fermion theory, we will say that a position j is occupied by a particle (hole) if $\phi_j\equiv-(Q+1)/2+j$ is included (not included) in the set $\{I_j\}$. Then the solution is classified by the particle-hole configurations. The ground state corresponds to the Fermi sea as shown in Fig. 2(a). An important class of levels is characterized by shifting the Fermi sea by m, i.e., choosing $I_j=\phi_j+m$. We call this the m-shifted levels and show in Fig. 2(b) an example. Creating particles and holes at either end of the m-shifted states generates the whole class of excited states which scale as $1/N$ in the critical phase. A general form of these excitations is obtained by creating n_p particles at positions $j=Q+m+p_k$ ($k=1,2,\ldots,n_p$), with $1\le p_1<p_2<\cdots<p_{n_p}$, n_h holes at $j=Q+m+1-h_k$, ($k=1,2,\ldots,n_h$), with $1\le h_1<h_2<\cdots<h_{n_h}$, $\bar n_p$ particles at $j=1+m-\bar p_k$ ($k=1,2,\ldots,\bar n_p$), with $1\le\bar p_1<\bar p_2<\cdots<\bar p_{\bar n_p}$, and $\bar n_h$ holes at $j=m+\bar h_k$ ($k=1,2,\ldots,\bar n_h$), with $1\le\bar h_1<\bar h_2<\cdots<\bar h_{\bar n_h}$. Without loss of generality we can set $n_p=n_h$ and $\bar n_p=\bar n_h$. A particle-hole configuration is collectively denoted by $\mathscr{P}$. And we will denote such an excited level by $(Q,m,\mathscr{P})$. In Figs. 2(c) and 2(d), we give some examples of particle-hole configurations.

For each range of Δ, there exists a variable transformation $p=p(\alpha)$ with which $\Theta^0(p(\alpha),p(\beta))$ defined in Eq. (17) depends only on $(\alpha-\beta)$ [3,10,23]. The resulting function is denoted by Θ, i.e., $\Theta(\alpha-\beta)\equiv\Theta^0(p(\alpha),p(\beta))$. $Z_N(\alpha)$ and $\Phi_{R,L}(\alpha)$ are defined similarly. We use superscript 0 for functions of p and no superscript for functions of α. In the thermodynamic limit $N\to\infty$ with fixed q, the Bethe ansatz equation becomes an integral equation for the phase function under the assumption that the solutions of Eq. (18) lie densely on a smooth curve $\mathscr{C}$ in the complex-α plane

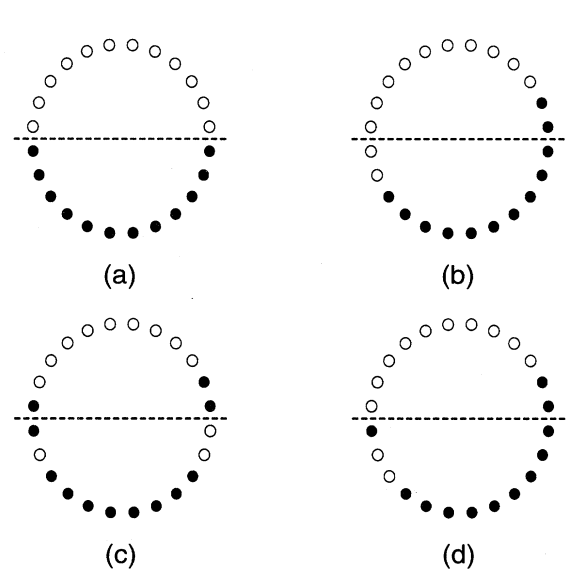

FIG. 2. A few examples of particle-hole configuration for $N=24$ and $Q=12$. We denote the values of I_j by the angular coordinate of equispaced small circles on a unit circle. Closed circles denote the occupied positions. (a) depicts the the ground state, (b) the $m=2$-shifted state, while (c) and (d) show excited states from (a) and (b), respectively.

with end points A and B. For the ground state, the contour is symmetric with respect to the imaginary axis so that $A=-B^*$. Then $Z_\infty\equiv\lim_{N\to\infty}Z_N$ should satisfy the integral equation

$$Z_\infty(\alpha)=p(\alpha)+2hi-\frac{1}{2\pi}\int_A^B\Theta(\alpha-\beta)Z'_\infty(\beta),d\beta, \qquad (24)$$

where $Z'_\infty(\beta)$ is the derivative of $Z_\infty(\beta)$ with respect to β. The solution of Eq. (24) depends on A and B which are determined as a function of q and h from the generalized normalization condition

$$Z_\infty(B)=\pi q \qquad (25)$$

or, equivalently,

$$Z_\infty(A)=-\pi q. \qquad (26)$$

Using the solution of Eq. (24) and relation (25), the bulk free energy e in Eq. (22) is given by

$$e(h,q)=\min_{R,L}\left\{\mp h\mp\frac{\delta}{2}-\frac{1}{2\pi}\int_A^B\Phi_{R,L}(\alpha)Z'_\infty(\alpha)d\alpha\right\}, \qquad (27)$$

where the upper (lower) sign corresponds to the $R(L)$ case.

Before closing this section, we discuss the effect of the twisted boundary condition. The boundary conditions do not affect the bulk properties but change the operator content and hence the TPF. The twisted boundary condition in the context of the XXZ chain is to impose the condition $\sigma^\pm_{N+1}=\exp(\pm 2\pi i l)\sigma^\pm_1$ where $\sigma^\pm_i$ are the Pauli spin operators. In the vertex model language, it is equivalent to assigning an extra vertex weight $\exp(\pm \pi i l)$ to the horizontal arrow in the first column. More generally, one can impose the twisted boundary conditions (l,l') by introducing the seams in the first column and the first row where an extra weight $e^{-i\pi l}(e^{i\pi l})$ is assigned to each right (left) arrow in the first column and similarly an extra weight $e^{i\pi l'}(e^{-i\pi l'})$ is assigned to each up (down) arrow in the first row. Note that the effect of the vertical field is to give each up (down) arrow on the lattice an extra weight e^{v} (e^{-v}). However, since the number of up arrows is conserved from row to row, one obtains the same effect if a vertical field of strength Mv is applied only to the first row of vertical bonds. Conversely, having a seam with extra weights $e^{\pm i\pi l'}$ is equivalent to assigning extra weights $e^{\pm i\pi l'/M}$ to all vertical bonds. A similar observation holds for the horizontal field also. Therefore one then sees that the ASSV model with the twisted boundary conditions (l,l') is equivalent to the ASSV model with periodic boundary conditions and with fields $\tilde{h}=h-\pi il/N$ and $\tilde{v}=v+\pi il'/M$. In the XXZ chain, such relations are achieved by a unitary transformation as discussed recently in [24]. A similar symmetry operation exists in the six-vertex model too [25].

III. SUMMATION FORMULA

In this section we derive the summation formula which converts the sum of the type

$$\mathscr{S}[f^0]=\frac{1}{N}\sum_{j=1}^{Q} f^0(p_j),$$

where f^0 is an arbitrary function which does not depend explicitly on h and q and $\{p_j\}$ is the solution of Eq. (18) for the $(Q,m,\mathscr{P})$ level, into a series in $1/N$. We obtain the expressions for the expansion coefficients and derive the useful relations between the correction terms and partial derivatives of the bulk contribution of the sum. The phase function in Eq. (16) and the energy eigenvalue in Eq. (20) involve such sums. For example,

$$E_Q^{R,L}=-N\left[\pm\left(h+\frac{\delta}{2}\right)+\mathscr{S}[\Phi^0_{R,L}]\right]. \tag{28}$$

With the change of variable $p=p(\alpha)$ explained in Sec. II, the sum becomes

$$\mathscr{S}[f]=\frac{1}{N}\sum_{j=1}^{Q} f(\alpha_j), \tag{29}$$

where $f(\alpha)\equiv f^0(p(\alpha))$ is not to be confused with the free energy $f(h,v)$, α_j are given by

$$\alpha_j=Z_N^{-1}\left(\frac{2\pi I_j}{N}\right), \tag{30}$$

with $\{I_j\}$ corresponding to the level $(Q,m,\mathscr{P})$, and Z_N^{-1} is the inverse function of Z_N. Using Eq. (30) in Eq. (29) and making explicitly reference to the locations of the particles, one can rewrite Eq. (29) as

$$\begin{aligned}\mathscr{S}[f]=&\frac{1}{N}\sum_{j=1}^{Q} f(Z_N^{-1}(\phi_j))+\frac{1}{N}\sum_{j=1}^{m}[f(Z_N^{-1}(\phi_{Q+j}))-f(Z_N^{-1}(\phi_j))]+\frac{1}{N}\sum_{k=1}^{n_p}[f(Z_N^{-1}(\phi_{Q+m+p_k}))-f(Z_N^{-1}(\phi_{Q+m+1-h_k}))]\\&+\frac{1}{N}\sum_{k=1}^{\bar{n}_p}[f(Z_N^{-1}(\phi_{1+m-\bar{p}_k}))-f(Z_N^{-1}(\phi_{m+\bar{h}_k}))],\end{aligned} \tag{31}$$

where

$$\phi_j=-\frac{Q+1}{2}+j. \tag{32}$$

The first sum is over the Fermi sea, the second accounts for the shift of the Fermi sea, and the third (fourth) accounts for the particle-hole configurations at the right (left) end of the Fermi sea. The general strategy here is to regard Z_N^{-1} as known first and determine it self-consistently later. To proceed, we make the crucial assumption that $Z_N^{-1}(\phi)$ has the finite first derivative at $\phi=\pm\pi q$. We will see later that the $1/N$ scaling of the energy gaps, or the mass gaps, throughout the critical phase follows from this assumption. Conversely, we assume here that the critical phase is characterized by the fact that $Z_N^{-1\prime}(\pm\pi q)$ exists with possible exceptions at some special points such as at $(h=0,\ q=1/2)$. Note that this assumption fails on the stochastic line $\tilde{\Delta}=1$ where the energy gap does not scale as $1/N$ but as $1/N^{3/2}$ [6]. Now, applying the Euler-Maclaurin sum formula to the first sum of Eq. (31) and using the Taylor expansion of $f(Z_N^{-1}(\phi))$ at $\phi=\pm\pi q$ for the rest of the sums, one then obtains, to $O(1/N^2)$,

$$\begin{aligned}\mathscr{S}[f]=&\frac{1}{2\pi}\int_{-\pi q}^{\pi q} f(Z_N^{-1}(\phi))d\phi+\frac{m}{N}[f(Z_N^{-1}(\phi))]\Big|_{-\pi q}^{\pi q}\\&+\frac{2\pi}{N^2}\left(\frac{m^2}{2}-\frac{1}{24}+\mathscr{N}\right)f'(Z_N^{-1}(\pi q))Z_N^{-1\prime}(\pi q)\\&-\frac{2\pi}{N^2}\left(\frac{m^2}{2}-\frac{1}{24}+\bar{\mathscr{N}}\right)f'(Z_N^{-1}(-\pi q))Z_N^{-1\prime}(-\pi q)\\&+o\left(\frac{1}{N^2}\right),\end{aligned} \tag{33}$$

where

$$\mathcal{N}=\sum_{k=1}^{n_p}(p_k+h_k-1),$$

$$\bar{\mathcal{N}}=\sum_{k=1}^{\bar{n}_p}(\bar{p}_k+\bar{h}_k-1). \tag{34}$$

This is not a correct $1/N$ expansion yet since $Z_N^{-1}(\phi)$ also has N dependence. We next assume that $Z_N(\alpha)$ possesses the $1/N$ expansion of the form

$$Z_N(\alpha)=Z_\infty(\alpha)+\frac{b_1(\alpha)}{N}+\frac{b_2(\alpha)}{N^2}+o\left(\frac{1}{N^2}\right), \tag{35}$$

where Z_∞, b_1, and b_2 should be determined later. Inverting Eq. (35) using the change of variable $\phi=Z_\infty(\alpha)$, the N dependence of $Z_N^{-1}(\phi)$ can be shown to be

$$Z_N^{-1}(\phi)=\alpha-\frac{1}{N}\frac{b_1(\alpha)}{Z'_\infty(\alpha)}+\frac{1}{N^2}\left[-\frac{b_2(\alpha)}{Z'_\infty(\alpha)}+\frac{1}{Z'_\infty(\alpha)}\left(\frac{b_1^2(\alpha)}{2Z'_\infty(\alpha)}\right)'\right]+o\left(\frac{1}{N^2}\right), \tag{36}$$

where $'$ denotes the derivative with respect to α. Inserting Eq. (36) into Eq. (33), expanding all quantities to $O(1/N^2)$ again, and changing the integration variable from ϕ to α, one finally obtains the summation formula in the form

$$\mathcal{S}[f]=\mathcal{S}_0+\frac{\mathcal{S}_1}{N}+\frac{\mathcal{S}_2}{N^2}+o\left(\frac{1}{N^2}\right), \tag{37a}$$

where

$$\mathcal{S}_0=\frac{1}{2\pi}\int_A^B f(\alpha)Z'_\infty(\alpha)d\alpha, \tag{37b}$$

$$\mathcal{S}_1=\frac{1}{2\pi}\int_A^B f'(\alpha)[2\pi m-b_1(\alpha)]d\alpha, \tag{37c}$$

$$\mathcal{S}_2=-\frac{1}{2\pi}\int_A^B f'(\alpha)b_2(\alpha)d\alpha+\frac{2\pi f'(B)}{Z'_\infty(B)}\left[\frac{m^2}{2}\left(1-\frac{b_1(B)}{2\pi m}\right)^2-\frac{1}{24}+\mathcal{N}\right]-\frac{2\pi f'(A)}{Z'_\infty(A)}\left[\frac{m^2}{2}\left(1-\frac{b_1(A)}{2\pi m}\right)^2-\frac{1}{24}+\bar{\mathcal{N}}\right]. \tag{37d}$$

Here A and $B=-A^*$ are determined from Eq. (25) or Eq. (26) as functions of h and q.

In deriving Eq. (37), we assumed the scaling form of Z_N in Eq. (35). The self-consistent equations for Z_∞, b_1, and b_2 are obtained by applying the summation formula to Eq. (16) with $f(\beta)=-\Theta(\alpha-\beta)$. Equating corresponding orders in both sides of the resulting equation, we obtain Eq. (24) for $Z_\infty(\alpha)$ and

$$\mathcal{T}\circ[b_1(\alpha)]=m[\Theta(\alpha-A)-\Theta(\alpha-B)], \tag{38a}$$

$$\mathcal{T}\circ[b_2(\alpha)]=\frac{2\pi}{Z'_\infty(B)}\left[\frac{m^2}{2}\left(1-\frac{b_1(B)}{2\pi m}\right)^2-\frac{1}{24}+\mathcal{N}\right]\times K(\alpha-B)-\frac{2\pi}{Z'_\infty(A)}\left[\frac{m^2}{2}\left(1-\frac{b_1(A)}{2\pi m}\right)^2-\frac{1}{24}+\bar{\mathcal{N}}\right]K(\alpha-A), \tag{38b}$$

where $K(\gamma)\equiv d\Theta(\gamma)/d\gamma$ and $\mathcal{T}$ is a linear operator defined by

$$\mathcal{T}\circ[G(\alpha)]\equiv G(\alpha)+\frac{1}{2\pi}\int_A^B K(\alpha-\beta)G(\beta)d\beta \tag{39}$$

for any function $G(\alpha)$.

The solutions of Eq. (38b) are written in terms of the function $F(\alpha,\mu,A,B)$ [14] which is defined by

$$\mathcal{T}\circ[F(\alpha,\mu,A,B)]=-\frac{1}{2}\Theta(\alpha-\mu). \tag{40}$$

Because Θ is odd and $A=-B^*$, it satisfies the relation

$$F(\alpha,\mu,A,B)=-F(-\alpha^*,-\mu^*,A,B). \tag{41}$$

Using the linearity of Eq. (38b), $b_1(\alpha)$ can be written as

$$b_1(\alpha)=2\pi m[1-D(\alpha)], \tag{42}$$

where

$$D(\alpha)=1-\frac{1}{\pi}[F(\alpha,B,A,B)-F(\alpha,A,A,B)], \tag{43}$$

the dressed charge function [14]. Equation (41) implies that $D(\alpha)=D(-\alpha^*)$ and so $D(A)=D(B)$. Using this, $b_2(\alpha)$ is written as

$$b_2(\alpha)=-\frac{4\pi}{Z'_\infty(B)}\left(\frac{D_0m^2}{2}-\frac{1}{24}+\mathcal{N}\right)\mathring{F}(\alpha,B,A,B)+\frac{4\pi}{Z'_\infty(A)}\left(\frac{D_0m^2}{2}-\frac{1}{24}+\bar{\mathcal{N}}\right)\mathring{F}(\alpha,A,A,B), \tag{44}$$

where $\mathring{F}(\alpha,\mu,A,B)\equiv-\partial F(\alpha,\mu,A,B)/\partial\mu$ and

$$D_0\equiv D(A)=D(B). \tag{45}$$

Inserting these into Eqs. (37c) and (37d), we can put $\mathcal{S}_1$ and $\mathcal{S}_2$ in the form

$$\mathcal{S}_1=m\int_A^B f'(\alpha)D(\alpha)d\alpha, \tag{46a}$$

$$\mathscr{S}_2=2\pi i\left(\frac{D_0^2m^2}{2}+\mathscr{N}-\frac{1}{24}\right)\zeta-2\pi i\left(\frac{D_0^2m^2}{2}+\bar{\mathscr{N}}-\frac{1}{24}\right)\bar{\zeta}, \tag{46b}$$

where

$$\zeta=\frac{1}{\pi i Z_\infty'(B)}\left(\int_A^B f'(\alpha)\mathring{F}(\alpha,B,A,B)d\alpha+\pi f'(B)\right), \tag{47a}$$

$$\bar{\zeta}=\frac{1}{\pi i Z_\infty'(A)}\left(\int_A^B f'(\alpha)\mathring{F}(\alpha,A,A,B)d\alpha+\pi f'(A)\right). \tag{47b}$$

We have used the notation ζ and $\bar{\zeta}$ for the quantities on the right-hand side of Eqs. (47a) and (47b), respectively, anticipating identification of them as the anisotropy factor. The fact that $\bar{\zeta}$ is the complex conjugate of ζ is not transparent in this form but will turn out to be the case, as will be seen later.

The resulting expressions for $\mathscr{S}_1$ and $\mathscr{S}_2$ seem to be rather complicated. But, the following manipulations show that they are related to the partial derivatives of $\mathscr{S}_0$ with respective to q and h. To take necessary derivatives, one needs to consider the variations of A , B, and $Z_\infty(\alpha)$, which is denoted as δA, δB, and $\delta Z_\infty(\alpha)$, respectively, upon the variations of h and q, denoted by δh and δq, respectively. We show in Appendix A that they are given by

$$Z_\infty'(A)\,\delta A=-[\pi+D_2(A)]\delta q-2iD_0\,\delta h, \tag{48a}$$

$$Z_\infty'(B)\,\delta B=[\pi-D_2(B)]\delta q-2iD_0\,\delta h, \tag{48b}$$

$$\delta Z_\infty(\alpha)=D_2(\alpha)\,\delta q+2iD(\alpha)\,\delta h, \tag{48c}$$

where

$$D_2(\alpha)\equiv F(\alpha,A,A,B)+F(\alpha,B,A,B). \tag{49}$$

From Eq. (41), one can see that $D_2(\alpha)=-D_2(-\alpha^*)$, which implies that $D_2(A)=-D_2(B)$. On the other hand, the variation of $\mathscr{S}_0$ given by Eq. (37b) is

$$\delta\mathscr{S}_0=\frac{1}{2}[f(A)+f(B)]\delta q-\frac{1}{2\pi}\int_A^B f'(\alpha)\,\delta Z_\infty(\alpha)d\alpha, \tag{50}$$

where a partial integration and Eqs. (25) and (26) are used. Combining Eqs. (50) and (48c), one obtains

$$\frac{\partial\mathscr{S}_0}{\partial h}=-\frac{i}{\pi}\int_A^B f'(\alpha)D(\alpha)d\alpha, \tag{51}$$

$$\frac{\partial\mathscr{S}_0}{\partial q}=\frac{1}{2}[f(A)+f(B)]-\frac{1}{2\pi}\int_A^B f'(\alpha)D_2(\alpha)d\alpha, \tag{52}$$

which, compared with Eq. (46a), gives the first relation

$$\mathscr{S}_1=i\pi m\frac{\partial\mathscr{S}_0}{\partial h}. \tag{53}$$

It means that the first-order correction term is proportional to the partial derivative of the bulk term with respect to h. Next, the variations of $\partial\mathscr{S}_0/\partial h$ and $\partial\mathscr{S}_0/\partial q$ are shown in Appendix A to take the form

$$\delta\left(\frac{\partial\mathscr{S}_0}{\partial h}\right)=J(\zeta+\bar{\zeta})\,\delta q-\frac{2iD_0^2}{\pi}(\zeta-\bar{\zeta})\,\delta h, \tag{54}$$

$$\delta\left(\frac{\partial\mathscr{S}_0}{\partial q}\right)=\frac{i\pi J^2}{2D_0^2}(\zeta-\bar{\zeta})\,\delta q+J(\zeta+\bar{\zeta})\,\delta h, \tag{55}$$

where ζ, $\bar{\zeta}$ are given in Eqs. (47a) and (47b), and

$$J\equiv D_0[1-D_2(B)/\pi]=D_0[1+D_2(A)/\pi]. \tag{56}$$

J is a constant depending on h, q, and Δ through A, B, and Θ. But we find that the value of J is equal to 1 for any values of A, and B whenever the function Θ is odd and $A=-B^*$. We give the detailed proof in Appendix B. Using the fact that $J=1$, we then have from Eq. (54) the expression for ζ and $\bar{\zeta}$ in terms of the partial derivatives of $\mathscr{S}_0$ as

$$\zeta=\frac{1}{2}\left[\frac{\partial}{\partial q}\left(\frac{\partial\mathscr{S}_0}{\partial h}\right)+\frac{\pi i}{2D_0^2}\frac{\partial^2\mathscr{S}_0}{\partial h^2}\right]=\frac{1}{2}\left[\frac{\partial}{\partial h}\left(\frac{\partial\mathscr{S}_0}{\partial q}\right)-\frac{2iD_0^2}{\pi}\frac{\partial^2\mathscr{S}_0}{\partial q^2}\right] \tag{57}$$

and

$$\bar{\zeta}=\frac{1}{2}\left[\frac{\partial}{\partial q}\left(\frac{\partial\mathscr{S}_0}{\partial h}\right)-\frac{\pi i}{2D_0^2}\frac{\partial^2\mathscr{S}_0}{\partial h^2}\right]=\frac{1}{2}\left[\frac{\partial}{\partial h}\left(\frac{\partial\mathscr{S}_0}{\partial q}\right)+\frac{2iD_0^2}{\pi}\frac{\partial^2\mathscr{S}_0}{\partial q^2}\right]. \tag{58}$$

Since $\mathscr{S}_0$ is a real function of h and q, it is now obvious that $\bar{\zeta}$ is indeed the complex conjugate of ζ as claimed before. From the second equalities in Eqs. (57) and (58), we find the identity

$$-\frac{\partial^2\mathscr{S}_0/\partial q^2}{\partial^2\mathscr{S}_0/\partial h^2}=\left(\frac{\pi}{2D_0^2}\right)^2. \tag{59}$$

This identity comes from the fact that $J=1$ and is very important in identifying the universality class of the critical phase of the ASSV model. Equations (53), (57), and (58) together with Eq. (46b) relate the finite-size corrections to the derivatives of $\mathscr{S}_0$. An analogous relation between the Fermi velocity and magnetic susceptibility has been obtained by Bogoliubov *et al.* [14] for the symmetric *XXZ* chain but our result is more general.

IV. FINITE-SIZE CORRECTIONS OF THE TRANSFER-MATRIX SPECTRA

In this section we investigate the finite-size corrections to the transfer-matrix spectra of the ASSV model using the summation formula obtained in Sec. III. For simplicity we only mention the case where $E_Q^R<E_Q^L$, which we call the R case. Most of the results are the same for both R and L

cases. The energy for the level $(Q,m,\mathscr{P})$ is obtained by applying the summation formula to Eq. (28):

$$E_{Q,m,\mathscr{P}}=-\left(h+\frac{\delta}{2}+\mathscr{S}_0[\Phi_R]\right)N-\mathscr{S}_1[\Phi_R]-\frac{\mathscr{S}_2[\Phi_R]}{N}+o\left(\frac{1}{N}\right), \tag{60}$$

where the leading-order term contributes to the bulk free energy, i.e., $\mathscr{S}_0=-e-h-\delta/2$ from Eq. (22). The correction term $\mathscr{S}_1$ is obtained from Eq. (53),

$$\mathscr{S}_1=-\pi i m(e_h+1), \tag{61}$$

and $\mathscr{S}_2$ is given by Eq. (46b) with

$$\zeta=-\frac{1}{2}e_{h,q}-\frac{\pi i}{4D_0^2}e_{h,h}=-\frac{1}{2}e_{q,h}+\frac{iD_0^2}{\pi}e_{q,q} \tag{62}$$

from Eq. (57). We use the shorthand notations that the subscripts of e denote partial derivatives. Note that $e_h=2\rho_L-1$ where ρ_L is the left-arrow density [for the L case, (e_h+1) in Eq. (61) is replaced by (e_h-1)].

The partition function is

$$\mathscr{Z}=\sum_{Q,m,\mathscr{P}}e^{-ME_{Q,m,\mathscr{P}}+vM(N-2Q)}, \tag{63}$$

with $E_{Q,m,\mathscr{P}}$ given in Eq. (60). To sum over the sectors Q, we take advantage of the fact that the summand in Eq. (63) is peaked around the value of Q near $\bar{Q}\equiv N\rho_D$. Inserting $q=\rho_D+(Q-\bar{Q})/N$ into Eq. (60) with a term $-v(1-2q)N$ added and expanding to order $1/N^2$ assuming $Q-\bar{Q}$ is of $O(1)$, one gets the scaling form of the energy as

$$E_{Q,m,\mathscr{P}}-v(1-2q)N=f(h,v)N+2\pi i m\rho_L-\frac{2\pi i\zeta'}{N}[m(Q-\bar{Q})+\mathscr{N}-\bar{\mathscr{N}}]+\frac{2\pi\zeta''}{N}\left(D_0^2m^2+\frac{e_{q,q}}{4\pi\zeta''}(Q-\bar{Q})^2+\mathscr{N}+\bar{\mathscr{N}}-\frac{1}{12}\right)+o\left(\frac{1}{N}\right), \tag{64}$$

where $f(h,v)$ is given in Eq. (23), ζ' (ζ'') is the real (imaginary) part of ζ given in Eq. (62), and the relation $\partial\rho_L/\partial q=e_{h,q}/2$ has been used.

On the other hand, for models whose TPF is given by Eq. (4), the $O(1/N)$ part of the energy is expected to behave as

$$\frac{2\pi\zeta''}{N}\left(\frac{m^2}{2g}+\frac{gn^2}{2}+\mathscr{N}+\bar{\mathscr{N}}-\frac{1}{12}\right)-\frac{2\pi i\zeta'}{N}(mn+\mathscr{N}-\bar{\mathscr{N}}).$$

Comparing Eq. (64) with this expression and identifying the indices m and $Q-\bar{Q}$ in Eq. (64) as the spin-wave and vortex indices, respectively, as in the SSV model, one sees that the critical phase of the ASSV model is indeed in the Gaussian-model universality class with $c=1$. The Gaussian coupling constant g can be read off from the coefficient of m^2 in Eq. (64) as

$$g=\frac{1}{2D_0^2}, \tag{65}$$

where we recall that D_0 is defined in Eqs. (45) and (43). However, from the coefficient of $(Q-\bar{Q})^2$ in Eq. (64), we also have the relation

$$g=\frac{e_{q,q}}{2\pi\zeta''}. \tag{66}$$

Therefore, to identify the spectra Eq. (64) with those of the Gaussian model, we require that the two expressions of g, Eqs. (65) and (66), give the identical result. Using Eqs. (62) and (65), the condition is then

$$-\frac{e_{q,q}}{e_{h,h}}=(\pi g)^2, \tag{67}$$

which is guaranteed by the identity (59). That the coupling constant g is given by Eq. (65) was derived by Izergin and Korepin [26] for the symmetric XXZ chain in a magnetic field but it also holds for the ASSV model. Using the relations

$$e_{h,h}=\frac{f_{h,h}f_{v,v}-f_{h,v}f_{v,h}}{f_{v,v}},$$

$$e_{q,q}=-\frac{4}{f_{v,v}}$$

in Eq. (67), we obtain an alternative expression given in Eq. (7).

Before proceeding further, we should question the validity of the summation formula. If the summation formula is valid with nontrivial solution for b_2, i.e., $b_2(\alpha)\neq 0$, then transfer-matrix spectra follow the scaling form in Eq. (1) which holds only in critical phase. We have assumed the expansion forms of Z_N and Z_N^{-1} in Eqs. (35) and (36) in deriving the summation formula. The assumption is valid only when the coefficients are finite, i.e., Z'_∞ should not be 0. Otherwise, the summation formula does not hold and the criticality is not guaranteed. So it is the sufficient condition for the criticality that $Z'_\infty(\alpha)$ and $b_2(\alpha)$ are not zero. It is compatible with earlier studies on the phase diagram of the asymmetric six-vertex model. First, for $\Delta\geq 1$ $Z'_\infty(A)$ is always 0 for any values of q when $\tilde{\Delta}=1$ [3]. At this point, the ASSV model describes stochastic growth models and the finite-size corrections show the Kardar-Parisi-Zhang scaling [4,6]. Another case occurs when $\Delta<-1$. In this region, the antiferroelectric phase exists when $q=1/2$. The phase boundary is given in [10] and $Z'_\infty(A)$ at the phase boundary is also 0. The finite-size corrections on this phase boundary have recently been studied by Albertini *et al.* [27]. There is also an ordered ferroelectric phase with $q=0$ in which case $A=B$ from Eqs. (25) and (26). In this case $b_2(\alpha)$ is identically zero and the system is also out of criticality. The $q=0$ and 1/2 ordered phases are separated from the critical phase by the Pokrovsky-Talapov (PT) transition [1].

Inside the phase boundaries stated above, the system is critical and the finite-size corrections to the transfer-matrix spectra are obtained from Eqs. (60), (61), (46b), and (62). Actual values of g through the critical phase can be mapped out without difficulty. In Fig. 3 we present the constant-g lines in the $(\tilde{\Delta},q)$ plane for $\tanh(2h)=0$, 0.5, and 1.0. As seen from Eq. (65), g is obtained from the function D. Instead of solving Eq. (40) for F and using Eq. (43), we study the integral equation for $D(\alpha)$,

$$\mathscr{T}\circ[D(\alpha)]=1, \tag{68}$$

which is obtained by combining Eqs. (43) and (40). It can be solved analytically only for special cases. For $q=0$, there is a trivial solution $D_0(\alpha)=1$. So we have $g=1/2$ at $q=0$ and at $q=1$ by symmetry. For $\Delta(=-\cosh\lambda)<-1$, the antiferroelectrically ordered phase with $q=1/2$ appears for $\tilde{\Delta}<\tilde{\Delta}_c$ with

$$\tilde{\Delta}_c\equiv\frac{\Delta}{\cosh(2h_c)}, \tag{69}$$

where h_c is given by [10]

$$h_c=\ln\cosh\lambda-\frac{\lambda}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-2n\lambda}\tanh(\lambda n). \tag{70}$$

The value of $\tilde{\Delta}_c$ decreases from -1 for $\tanh(2h)=0$ to -4 for $\tanh(2h)=\pm1$. At the phase boundary, the value of B is $\pi+i\lambda$ [10]. Then Eq. (68) is solved by Fourier series method to yield the solution $D(\alpha)=1/2$. So we have $g=2$ at the antiferroelectric phase boundary. For $\Delta>1$, the ASSV model is critical only for $\tilde{\Delta}<1$. As $\tilde{\Delta}$ approaches 1 from below, g decreases to 0. For the SSV model, the model is critical for $-1<\Delta(=-\cos\gamma)<1$ and the value of g is given by the simple formula $g=1-\gamma/\pi$ [13]. In this case, $B\to\infty$ and naive application of Eq. (68) is invalid due to the fact that $Z'_\infty(\pm\infty)=0$. However, a careful treatment using the Wiener-Hopf method can reproduce the correct result starting from Eq. (68) [14,25]. For other cases, the values of g are obtained by solving the integral equation (68) numerically. Note that there is a discontinuity in g from 1 to 2 at $(\tilde{\Delta},q)=(-1,1/2)$ for $\tanh(2h)=0$. It is the Kosterlitz-Thouless (KT) transition point where the free energy has the essential singularity [23]. There is a crossover from the KT transition to the PT transition for $\tanh(2h)\neq0$.

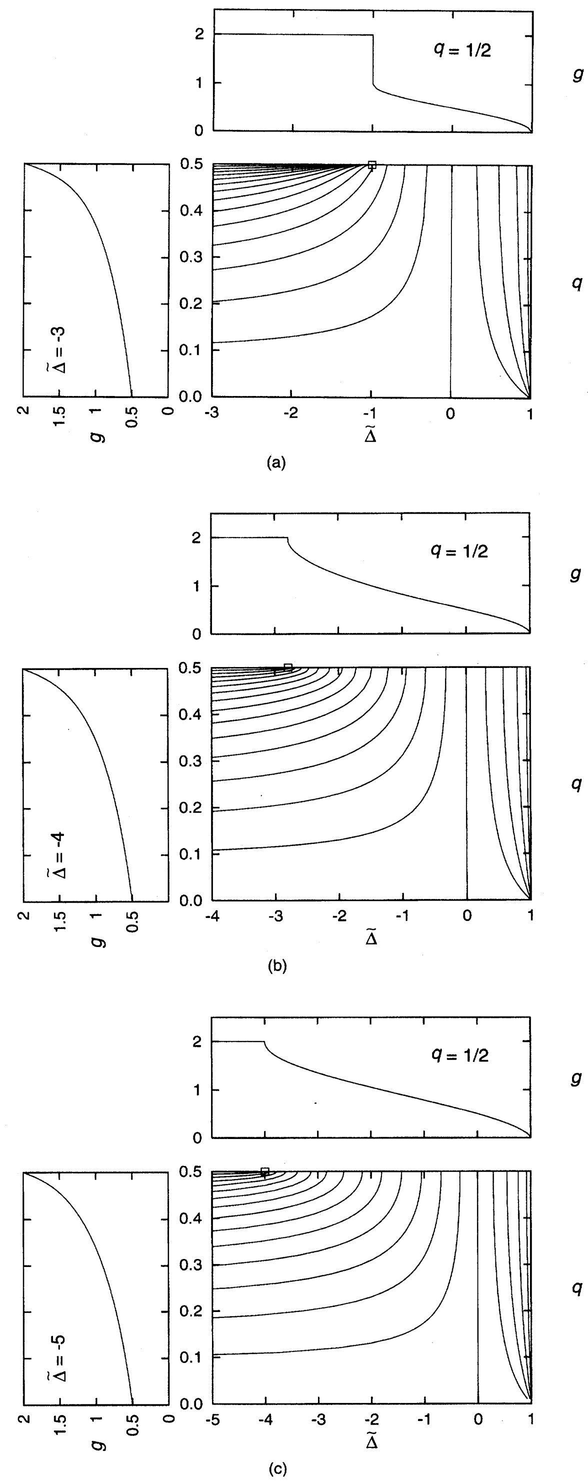

FIG. 3. The constant-g lines are plotted in the $(\tilde{\Delta},q)$ plane for $\tanh(2h)=0$ in (a), 0.5 in (b), and 1 in (c). These are obtained from the numerical solutions of Eq. (68). On $\tilde{\Delta}=0$ and $q=0$ lines, $g=1/2$. On $\tilde{\Delta}=1$ line $g=0$. The value of g increases in steps of 0.1 from right to left and reaches 2 at $q=1/2$ and $\tilde{\Delta}<\tilde{\Delta}_c$. The figures at the top show g as a function of $\tilde{\Delta}$ at $q=1/2$ and the figures at left show g as a function of q at $\tilde{\Delta}=-3$ in (a), -4 in (b), and -5 in (c). The rectangular symbols denote the phase-transition points $\tilde{\Delta}_c$ from the critical phase to the $q=1/2$ ordered phase.

We next derive the TPF $\tilde{\mathscr{Z}}$. Using the scaling form of Eq. (64) and the identity Eq. (67), it can be put in the form

$$\tilde{\mathscr{Z}}=\sum_{Q,m\in\mathbf{Z}}\sum_{\mathscr{P}} e^{-2\pi i m\rho_L M}\exp\left[2\pi i\tau\left(\Delta_{m,Q-\bar{Q}}+\mathscr{N}-\frac{1}{24}\right)\right.$$
$$\left.-2\pi i\bar{\tau}\left(\bar{\Delta}_{m,Q-\bar{Q}}+\bar{\mathscr{N}}-\frac{1}{24}\right)\right], \qquad (71)$$

where the modular ratio τ is given as $\tau=(M/N)\zeta$. The sum over $\mathscr{P}$ produces the factor

$$\sum_{\mathscr{P}} \mathsf{q}^{\mathscr{N}-1/24}\,\bar{\mathsf{q}}^{\bar{\mathscr{N}}-1/24}=\frac{1}{\eta(\mathsf{q})\,\eta(\bar{\mathsf{q}})},$$

where the nome $\mathsf{q}=e^{2\pi i\tau}$ and $\eta(\mathsf{q})$ is the Dedekind eta function given in Eq. (5). Now we introduce the two mismatch parameters

$$\alpha=\{\rho_L M\} \text{ and } \beta=\{\rho_D N\}=\{\bar{Q}\}, \qquad (72)$$

where $\{x\}$ denotes the fractional part of x; i.e., α and β are the mean number of left arrows per column and down arrows per row, respectively, modulo 1. We next change the index Q in Eq. (71) to n by the relation $Q-\bar{Q}=n-\beta$. Then the TPF for the critical phase of the ASSV model is given in the form of the modified Coulombic partition function as given in Eq. (8). The mismatches come from the incommensuration of arrow densities ρ_L and ρ_D with system size M and N, respectively.

We also consider the effect of the twisted boundary condition on the TPF. Since the twisted boundary condition (l,l') corresponds to the ASSV model with the periodic boundary conditions but with the modified fields $h-\pi il/N$ and $v+\pi il'/M$, these $O(1/N)$ changes of h and v modify the TPF. Thus one replaces h and v by $h-\pi il/N$ and $v+\pi il'/M$, respectively, in Eq. (64), and makes Taylor expansion to the necessary order in $1/N$ to obtain the boundary condition effect on the finite-size corrections. After a straightforward calculation the resulting TPF is found to be generalized to

$$\tilde{\mathscr{Z}}(\rho_L,\rho_D;l,l';M,N|\tau)=e^{\pi iMl(2\rho_L-1)-\pi iNl'(2\rho_D-1)}$$
$$\times\tilde{\mathscr{Z}}_{\mathrm{gC}}(\{M\rho_L\},\{N\rho_D\};l,l'|\tau), \qquad (73)$$

where $\tilde{\mathscr{Z}}_{\mathrm{gC}}$ is the generalized Coulombic partition function defined as

$$\tilde{\mathscr{Z}}_{\mathrm{gC}}(\alpha,\beta;l,l'|\tau)\equiv\frac{1}{|\eta(\mathsf{q})|^2}\sum_{m,n\in\mathbf{Z}} e^{-2\pi i m\alpha-2\pi i l'(n-\beta)}$$
$$\times\mathsf{q}^{\Delta_{m-l,n-\beta}}\bar{\mathsf{q}}^{\bar{\Delta}_{m-l,n-\beta}}. \qquad (74)$$

The toroidal partition functions in Eqs. (8) and (73) satisfy the necessary modular covariance as they should. These are discussed in Appendix C.

V. DISCUSSIONS

In this paper we utilized a method of calculating the finite-size corrections applicable to the critical phase of the ASSV model and investigated the FSS of the transfer-matrix spectra of the model for arbitrary sector $Q=qN$. Since we work for general q, the string solutions appearing only at $q=1/2$ do not complicate the analysis [28] and low-lying excitations are easily classified. For any low-lying levels, the finite-size corrections for arbitrary sum over functions of fugacities are related to partial derivatives of the bulk contributions; Eq. (53) for the first-order term and Eqs. (46b), (57), and (58) for the second-order term. From this, finite-size scaling amplitudes of the ASSV model are related to thermodynamic quantities. Those relations are derived using only the algebraic structures of the Bethe ansatz equation and the form of the energy function is irrelevant to the FSS properties as long as it does not depend explicitly on h and q. In particular, the energy of the asymmetric XXZ chain [6] is given as

$$E=\frac{1}{\cosh(2h)}\sum_{j=1}^{Q}(\Delta-\cos p_j), \qquad (75)$$

where $\{p_j\}$ satisfy the Bethe ansatz equation (18). If we consider $(\cosh 2h)E$ rather than the energy itself, the summand in Eq. (75) does not explicitly depend on h and q and the results of Secs. III and IV are directly applicable.

The ASSV model is conveniently parametrized by the interaction parameter $\tilde{\Delta}$ [Eq. (11)], the anisotropy δ [Eq. (9)], the horizontal field h, and finally vertical field v or alternatively, the down-arrow density q. For fixed h, δ, and $q(\neq 1/2)$, the model is critical for $-\infty<\tilde{\Delta}<1$. When $q=1/2$, however, the region defined by $\tilde{\Delta}<\tilde{\Delta}_c$ with $\tilde{\Delta}_c$ given by Eq. (69) corresponds to the antiferroelectrically ordered phase. We showed that the critical phase is in the Gaussian-model universality class with $c=1$ and the Gaussian coupling constant g as given by Eq. (65). The value of g ranges from 0 at the stochastic limit $\tilde{\Delta}=1$ to 1/2 at the PT transition lines bordering the $q=0$ or $q=1$ phases and to 2 at the PT transition lines bordering the $q=1/2$ antiferroelectrically ordered phase.

We also constructed the TPF in Eq. (8) for periodic boundary conditions and in Eq. (73) for more general twisted boundary conditions. Due to the incommensuration of arrow densities, it takes an additional phase factor $e^{-2\pi i m\alpha}$ and a constant shift β in the vortex-excitation index n. The phase factor comes from the $O(1)$ imaginary term of the energy spectra Eq. (61) which can be understood as follows. The continuous lines connecting the left pointing arrows may be pictured as domain walls or steps running across the lattice. Since there are $\rho_L M$ of them, the mean distance between them is $1/\rho_L$ lattice units. Thus the periodicity of the lattice in the time direction is enlarged by the same factor and the transfer matrix $\mathbf{T}$ is the ρ_Lth power of the time translation operator for one unit distance. Therefore a factor $\exp(2\pi i\rho_L)$ may appear in the spectrum. As shown in Appendix C, the mismatch parameter in the phase factor of Eq. (8) becomes that which shifts the vortex-excitation index upon the modular transformation which simply rotates the lattice by 90°, exchanging the role of M and N.

Our result for the TPF is consistent with all other previous results. For the five-vertex model in the noninteracting case, the TPF on a deformed square lattice is given as [7]

$$\tilde{\mathscr{Z}}_{5\text{-}v}=\frac{1}{|\eta|^2}\sum_{m,n\in\mathbf{Z}} e^{-\pi i Q_1 m}\mathsf{q}^{\Delta_{m,n-Q_0}}\bar{\mathsf{q}}^{\bar{\Delta}_{m,n-Q_0}}, \tag{76}$$

where $Q_0=(1-\rho_D)N$ and $Q_1=(2\rho_L-\rho_D-1)M$. The TFP's on different geometry can be transformed to each other through the modular transformation (see Appendix C). One can find that Eq. (76) is obtained from Eq. (8) through the modular transformation $\tau\rightarrow\tau+M/(2N)$ with $M/(2N)=1$. In the SSV model, $\rho_L=\rho_D=1/2$ and hence α (β) is 0 if M (N) is even and 1/2 if odd. If the twisted boundary condition $(l,0)$ is applied in the SSV model, the operator content of this model obtained by other methods is summarized by the TPF [22]

$$\tilde{\mathscr{Z}}=\frac{1}{|\eta(\mathsf{q})|^2}\sum_{m,n\in\mathbf{Z}} e^{2\pi i m\mu}\mathsf{q}^{\Delta_{m-l,n-\nu}}\bar{\mathsf{q}}^{\Delta_{m-l,n-\nu}}, \tag{77}$$

where $\mu(\nu)$ is 0 for $M(N)$ even and 1/2 for odd. It is in agreement with Eq. (73) when one uses $\rho_L=\rho_D=\frac{1}{2}$ and $l'=0$ in Eq. (73).

The asymmetric XXZ chain is obtained if one takes an extreme-anisotropic limit in which the vertex weight approaches the limit $w_3=w_4=0$ and $w_1=w_2=w_5=w_6=1$. With an appropriate parametrization, $\mathbf{T}$ can be put in the form

$$\mathbf{T}(u)=\exp[i\mathbf{P}-u\mathbf{H}_{XXZ}+O(u^2)], \tag{78}$$

where u is the so-called spectral parameter as a function of which $\mathbf{T}(u)$ form a commuting family, $\mathbf{P}$ is the shift operator, and $\mathbf{H}_{XXZ}$ is the asymmetric XXZ chain Hamiltonian. Therefore $\mathbf{T}$ and $\mathbf{H}_{XXZ}$ share a same set of spectra. However, the TPF is not the same as Tr $\exp(-uM\mathbf{H}_{XXZ})$. This is why complete information on the operator content for the XXZ chains is not sufficient to construct the TPF of the lattice model.

When the ASSV model is considered as a model for the ECS of, e.g., a fcc (110) surface, the free energy $f(h,v)$ itself is the height of the surface from the base (110) plane with appropriate identification of coordinates. In particular, the surface curvature κ defined in, e.g., [2], is related to the Hessian of f by

$$\kappa=\frac{d^2}{k_BT}H^{1/2}, \tag{79}$$

where d is the distance between the crystal planes. Our result, Eq. (7), then relates κ to g by

$$\kappa=\frac{2d^2}{\pi k_BT}\frac{1}{g} \tag{80}$$

exactly. When the interactions are intrinsically antiferroelectric, the $q=1/2$ phase at low temperatures corresponds to the flat (110) facet. As the temperature is raised the facet area becomes smaller and finally disappears at the roughening temperature T_R. The roughening transition corresponds to the KT point where $g=1$. Thus there occurs a universal jump in the curvature of magnitude $2d^2/(\pi k_BT_R)$. This universal jump has been anticipated from the Coulomb gas picture of the solid-on-solid models [29] and also from the solution of the six-vertex model near $h=v=0$ [2] and has been measured experimentally [30]. Our result goes beyond this. Equation (80) being exact, it applies to all points of the curved portion of the crystal. In particular, near the PT transitions where the curved surface joins the facet smoothly with the exponent $\theta=3/2$ [31], one expects the universal jump of the curvature with $g=2$ or $g=1/2$ depending on which portion of the PT line is appropriate. It is interesting to speculate whether the relation Eq. (80) also holds for other solid-on-solid type models and for real samples. If it is valid for real systems, it would provide a means to measure the critical exponent directly from surface curvatures.

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APPENDIX A: VARIATIONS OF THE PARTIAL DERIVATIVES OF $\mathscr{S}_0$

In this appendix we consider the variations of $\mathscr{S}_0$ upon the variations of h and q. As h and q vary, the end points A and B also vary, satisfying

$$\pi\,\delta q=\delta Z_\infty(B)+Z_\infty'(B)\,\delta B \tag{A1}$$

and

$$-\pi\,\delta q=\delta Z_\infty(A)+Z_\infty'(A)\,\delta A. \tag{A2}$$

They are obtained from the variations of Eqs. (25) and (26). Since this involves $\delta Z_\infty(\alpha)$, we take the variation of Eq. (24) after performing the partial integration inside the integral operator. This gives a simpler result,

$$\delta Z_\infty(\alpha)=2iD(\alpha)\,\delta h+D_2(\alpha)\,\delta q, \tag{A3}$$

with D and D_2 defined by Eqs. (43) and (49). In taking the variations in this appendix, it is convenient to use the integral representations of D and D_2 rather than the definitions in Eqs. (43) and (49). The integral representation for D is given in Eq. (68) and the other for D_2 are easily obtained by combining Eqs. (49) and (40):

$$\mathscr{T}^{\circ}[D_2(\alpha)]=-\frac{1}{2}[\Theta(\alpha-A)+\Theta(\alpha-B)], \tag{A4}$$

where the operator $\mathscr{T}$ is defined in Sec. III. Inserting Eq. (A3) into Eqs. (A1) and (A2) gives Eq. (48).

Next, consider the variation of Eqs. (51) and (52):

$$\delta\left(\frac{\partial\mathscr{S}_0}{\partial h}\right)=-\frac{i}{\pi}\left(\int_A^B f'(\alpha)\,\delta D(\alpha)\,d\alpha+f'(B)D_0\,\delta B\right.$$
$$\left.-f'(A)D_0\,\delta A\right), \tag{A5}$$

$$\delta\left(\frac{\partial \mathscr{S}_0}{\partial q}\right) = -\frac{1}{2\pi}\int_A^B f'(\alpha)\,\delta D_2(\alpha)\,d\alpha + \frac{f'(A)}{2}\left(1+\frac{D_2(A)}{\pi}\right)\delta A + \frac{f'(B)}{2}\left(1-\frac{D_2(B)}{\pi}\right)\delta B. \quad \text{(A6)}$$

The variations of $D(\alpha)$ and $D_2(\alpha)$ are obtained from Eqs. (68) and (A4). After a straightforward calculation, one can find that

$$\delta D(\alpha) = \frac{D_0}{\pi}\{\mathring{F}(\alpha,B,A,B)\,\delta B - \mathring{F}(\alpha,A,A,B)\,\delta A\}, \quad \text{(A7)}$$

$$\delta D_2(\alpha) = -\left(1-\frac{D_2(B)}{\pi}\right)\mathring{F}(\alpha,B,A,B)\,\delta B - \left(1+\frac{D_2(A)}{\pi}\right)\mathring{F}(\alpha,A,A,B)\,\delta A. \quad \text{(A8)}$$

Using Eqs. (A7) and (A8) in Eqs. (A5) and (A6) and using the definitions of ζ and $\bar{\zeta}$ in Eq. (47), one obtains

$$\delta\left(\frac{\partial \mathscr{S}_0}{\partial h}\right) = -\frac{iD_0}{\pi}[i\zeta Z'_\infty(B)\,\delta B - i\bar{\zeta} Z'_\infty(A)\,\delta A], \quad \text{(A9)}$$

$$\delta\left(\frac{\partial \mathscr{S}_0}{\partial q}\right) = \frac{1}{2}\left(1-\frac{D_2(B)}{\pi}\right) i\zeta Z'_\infty(B)\,\delta B + \frac{1}{2}\left(1+\frac{D_2(A)}{\pi}\right) i\bar{\zeta} Z'_\infty(A)\,\delta A, \quad \text{(A10)}$$

which together with Eq. (48) give Eqs. (54) and (55).

APPENDIX B: THE PROOF OF $J=1$

In this appendix we present some identities between the partial derivatives of $F(\alpha,\mu,A,B)$ defined in Eq. (40) and prove the identity $J \equiv D(B)[1-D_2(B)/\pi]=1$, where

$$D(B) = 1-\frac{1}{\pi}[F(B,B,A,B)-F(B,A,A,B)], \quad \text{(B1)}$$

$$D_2(B) = F(B,B,A,B)+F(B,A,A,B). \quad \text{(B2)}$$

In general, J may be a function of a and b with $A=-a+ib$ and $B=a+ib$. But, Eq. (40) is invariant under the shift of all arguments by an imaginary amount, i.e., $F(\alpha,\mu,A,B)=F(\alpha+iu,\mu+iu,A+iu,B+iu)$ for any real u. This means that $D(B)$, $D_2(B)$, and thus J are functions of only a. When $a=0$ the function $F(\alpha,\mu,A,B)$ becomes simply $-\frac{1}{2}\Theta(\alpha-\mu)$ and so $J=1$. Thus, for the proof of $J=1$, it suffices to show that the total derivative of J with respect to a is identically 0.

The total derivative of J contains the partial derivatives of $F(B,B,A,B)$ and $F(B,A,A,B)$ with respect to a:

$$\frac{d}{da}F(B,B,A,B) = F' - \mathring{F} - F_A + F_B, \quad \text{(B3)}$$

$$\frac{d}{da}F(B,A,A,B) = F' + \mathring{F} - F_A + F_B, \quad \text{(B4)}$$

where $F' \equiv \partial F/\partial\alpha$, $\mathring{F} \equiv -\partial F/\partial\mu$, $F_A \equiv \partial F/\partial A$, and $F_B \equiv \partial F/\partial B$. We use the convention that the arguments of the functions are (α,μ,A,B) when not shown explicitly. First, we derive several identities between partial derivatives of F, which will simplify Eqs. (B3) and (B4). Taking partial derivatives of Eq. (40), one obtains integral equations for F', F_A, and F_B. After some manipulations, they take the form

$$\mathscr{T}\circ F' = -\frac{1}{2}K(\alpha-\mu) + \frac{1}{2\pi}[F(B,\mu,A,B)K(\alpha-B) - F(A,\mu,A,B)K(\alpha-A)], \quad \text{(B5)}$$

$$\mathscr{T}\circ F_A = \frac{1}{2\pi}F(A,\mu,A,B)K(\alpha-A), \quad \text{(B6)}$$

$$\mathscr{T}\circ F_B = -\frac{1}{2\pi}F(B,\mu,A,B)K(\alpha-B), \quad \text{(B7)}$$

where $\mathscr{T}$ is defined in Eq. (39). Note that F and $\mathring{F}$ can be written as

$$F(\alpha,\mu,A,B) = \mathscr{T}^{-1}\circ[-\tfrac{1}{2}\Theta(\alpha-\mu)], \quad \text{(B8)}$$

$$\mathring{F}(\alpha,\mu,A,B) = \mathscr{T}^{-1}\circ[-\tfrac{1}{2}K(\alpha-\mu)], \quad \text{(B9)}$$

where $\mathscr{T}^{-1}$ is the inverse operator of $\mathscr{T}$. Using the linearity of Eqs. (B5), (B6), and (B7), F', F_A, and F_B can be obtained by applying $\mathscr{T}^{-1}$, to Eqs. (B5), (B6), and (B7) which yields

$$F'(\alpha,\mu,A,B) = \mathring{F}(\alpha,\mu,A,B) - \frac{1}{\pi}[F(B,\mu,A,B)\mathring{F}(\alpha,B,A,B) - F(A,\mu,A,B)\mathring{F}(\alpha,A,A,B)], \quad \text{(B10)}$$

$$F_A(\alpha,\mu,A,B) = -\frac{1}{\pi}F(A,\mu,A,B)\mathring{F}(\alpha,A,A,B), \quad \text{(B11)}$$

$$F_B(\alpha,\mu,A,B) = \frac{1}{\pi}F(B,\mu,A,B)\mathring{F}(\alpha,B,A,B). \quad \text{(B12)}$$

Using these relations in Eqs. (B3) and (B4), one can obtain

$$\frac{d}{da}F(B,B,A,B) = \frac{2}{\pi}F(A,B,A,B)\mathring{F}(B,A,A,B), \quad \text{(B13)}$$

$$\frac{d}{da}F(B,A,A,B) = 2\mathring{F}(B,A,A,B)$$

$$\times\left[1+\frac{1}{\pi}F(A,B,A,B)\mathring{F}(B,A,A,B)\right], \tag{B14}$$

from which the derivatives of $D(B)$ and $D_2(B)$ can be written as

$$\frac{dD(B)}{da}=\frac{2}{\pi}\mathring{F}(B,A,A,B)\times\left\{1+\frac{1}{\pi}[F(A,A,A,B)-F(A,B,A,B)]\right\}, \tag{B15}$$

$$\frac{dD_2(B)}{da}=2\mathring{F}(B,A,A,B)\times\left\{1+\frac{1}{\pi}[F(A,A,A,B)+F(A,B,A,B)]\right\}. \tag{B16}$$

As explained in Sec. III, $F(A,A,A,B)=-F(B,B,A,B)$ and $F(B,A,A,B)=-F(A,B,A,B)$ because the function Θ is odd and $A=-B^*$. Using these properties in Eqs. (B15) and (B16), one can find that

$$\frac{d}{da}D(B)=\frac{2}{\pi}\mathring{F}(B,A,A,B)D(B), \tag{B17}$$

$$\frac{d}{da}\left(1-\frac{D_2(B)}{\pi}\right)=-\frac{2}{\pi}\mathring{F}(B,A,A,B)\left(1-\frac{1}{\pi}D_2(B)\right). \tag{B18}$$

Then the total derivative of J is obtained straightforwardly.

$$\frac{dJ}{da}=\frac{dD(B)}{da}\left(1-\frac{D_2(B)}{\pi}\right)+D(B)\frac{d}{da}\left(1-\frac{D_2(B)}{\pi}\right)=0. \tag{B19}$$

Equation (B19) together with $J=1$ at $a=0$ implies that $J=1$ for any value of a.

APPENDIX C: MODULAR TRANSFORMATION PROPERTIES OF $\tilde{\mathscr{Z}}$

In this appendix we examine the modular covariance of the TPF for the ASSV model in Eq. (73). It is convenient to work with the Gaussian model TPF with the shift boundary conditions [12]

$$\mathscr{Z}_{m,n}(\tau)=\mathscr{Z}_0\exp\left[-\frac{\pi g}{\tau''}|n-\tau m|^2\right], \tag{C1}$$

where

$$\mathscr{Z}_0(\tau)=\frac{g}{\tau''}\frac{1}{\eta(\mathrm{q})\,\eta(\bar{\mathrm{q}})} \tag{C2}$$

is the Gaussian model TPF with the periodic boundary conditions. They have modular transformation properties as

$$\mathscr{Z}_{m,n}(\tau)=\mathscr{Z}_{m,n+m}(\tau+1)=\mathscr{Z}_{-n,m}\left(-\frac{1}{\tau}\right) \tag{C3}$$

or combining these

$$\mathscr{Z}_{m,n}(\tau)=\mathscr{Z}_{dm+cn,an+bm}\left(\frac{a\tau+b}{c\tau+d}\right), \tag{C4}$$

where $a,b,c,$ and d are integers with $ad-bc=1$.

The generalized Coulombic partition function defined in Eq. (74) is written in terms of $Z_{m,n}$ by applying the Poisson sum formula

$$\sum_{m=-\infty}^{\infty}f(m-l)=\sum_{m=-\infty}^{\infty}e^{2\pi iml}g(2\pi m), \tag{C5}$$

with

$$g(k)\equiv\int_{-\infty}^{\infty}dx\,f(x)e^{ikx}. \tag{C6}$$

First, Eq. (74) can be written as

$$\tilde{\mathscr{Z}}_{\mathrm{gC}}(\alpha,\beta;l,l'|\tau)=\frac{1}{|\eta(\mathrm{q})|^2}\sum_{n\in\mathbf{Z}}\exp[-\pi g\tau''(n-\beta)^2-2\pi il'(n-\beta)-2\pi i\alpha l]\times\sum_{m\in\mathbf{Z}}\exp\left[-\frac{\pi\tau''}{g}(m-l)^2+2\pi i[\tau'(n-\beta)-\alpha](m-l)\right]. \tag{C7}$$

Then, applying the Poisson sum formula to the sum over m in Eq. (C7), one can express $\tilde{\mathscr{Z}}_{\mathrm{gC}}$ in terms of $\mathscr{Z}_{n,m}$ as

$$\tilde{\mathscr{Z}}_{\mathrm{gC}}(\alpha,\beta;l,l'|\tau)=\mathscr{Z}_0(\tau)\sum_{m,n\in\mathbf{Z}}e^{-2\pi i(m+\alpha)l-2\pi il'(n-\beta)}\mathscr{Z}_{n-\beta,m+\alpha}(\tau). \tag{C8}$$

The TPF in Eq. (73) becomes

$$\tilde{\mathscr{Z}}(\rho_L,\rho_D;l,l';M,N|\tau)=\exp[\pi iMl(2\rho_L-1)-\pi iNl'(2\rho_D-1)]\mathscr{Z}_0(\tau)\sum_{m,n\in\mathbf{Z}}e^{-2\pi i(m+\alpha)l-2\pi il'(n-\beta)}\mathscr{Z}_{n-\beta,m+\alpha}(\tau), \tag{C9}$$

where $\alpha \equiv \{M\rho_L\}$ and $\beta \equiv \{N\rho_D\}$ where $\{x\}$ denotes the fractional part of x.

Under the transformation $\tau \rightarrow \tau+1$,

$$\begin{aligned}\tilde{\mathscr{Z}}_{\rm gC}(\alpha,\beta;l,l'|\tau) &= \mathscr{Z}_0(\tau)\sum_{m,n\in\mathbf{Z}} e^{-2\pi i l(m+\alpha)-2\pi i l'(n-\beta)}\mathscr{Z}_{n-\beta,m+n+\alpha-\beta}(\tau+1)\\ &= \mathscr{Z}_0(\tau+1)\sum_{m,n\in\mathbf{Z}} e^{-2\pi i l(m+\alpha-\beta)-2\pi i(l'-l)(n-\beta)}\mathscr{Z}_{n-\beta,m+\alpha-\beta}(\tau+1)\\ &= \tilde{\mathscr{Z}}_{\rm gC}(\alpha-\beta,\beta;l,l'-l|\tau+1), \end{aligned} \tag{C10}$$

which means that

$$\tilde{\mathscr{Z}}_{\rm gC}(\alpha,\beta;l,l'|\tau+1) = \tilde{\mathscr{Z}}_{\rm gC}(\alpha+\beta,\beta;l,l'+l|\tau). \tag{C11}$$

So, the TPF $\tilde{\mathscr{Z}}$ transforms under the transformation $\tau \rightarrow \tau+1$ as

$$\tilde{\mathscr{Z}}(\rho_L,\rho_D;l,l';M,N|\tau+1) = \tilde{\mathscr{Z}}(\rho_L',\rho_D;l,l'+l;M+N,N|\tau), \tag{C12}$$

where

$$\rho_L' = \frac{M\rho_L+N\rho_D}{M+N}. \tag{C13}$$

Similarly, $\tilde{\mathscr{Z}}_{\rm gC}$ transforms under the transformation $\tau \rightarrow -1/\tau$ as

$$\begin{aligned}\tilde{\mathscr{Z}}_{\rm gC}(\alpha,\beta;l,l'|\tau) &= \mathscr{Z}_0(\tau)\sum_{m,n\in\mathbf{Z}} e^{-2\pi i(m+\alpha)l-2\pi i l'(n-\beta)}\mathscr{Z}_{-m-\alpha,n-\beta}(-\tfrac{1}{\tau})\\ &= \mathscr{Z}_0\left(-\frac{1}{\tau}\right)\sum_{m,n\in\mathbf{Z}} e^{-2\pi i l'(n-\beta)-2\pi i(-l)(m-\alpha)}\mathscr{Z}_{m-\alpha,n-\beta}(-\tfrac{1}{\tau})\\ &= \tilde{\mathscr{Z}}_{\rm gC}(-\beta,\alpha;l',-l|-\tfrac{1}{\tau}), \end{aligned} \tag{C14}$$

which means that

$$\tilde{\mathscr{Z}}_{\rm gC}(\alpha,\beta;l,l'|-\tfrac{1}{\tau}) = \tilde{\mathscr{Z}}_{\rm gC}(\beta,-\alpha;-l',l|\tau). \tag{C15}$$

From this, one can also find that

$$\tilde{\mathscr{Z}}(\rho_L,\rho_D;l,l';M,N|-\tfrac{1}{\tau}) = \tilde{\mathscr{Z}}(\rho_L{}',\rho_D{}';-l',l;N,M|\tau), \tag{C16}$$

where

$$\rho_L' = \rho_D, \quad \rho_D' = 1-\rho_L. \tag{C17}$$

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